Holomorphic Dynamics - Lecture 3

$\mathbb{D} \cong \mathbb{H}$

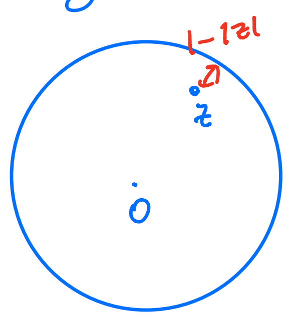
$\mathbb{C}$

$\hat{\mathbb{C}}$

Poincaré disk

Thm There exists a unique Riemannian metric on $\mathbb{D}$ which is invariant under $\mathrm{Aut}(\mathbb{D})$.

It is given by $ds = \dfrac{2\,|dz|}{1-|z|^2}$



Rmk if $|z| \sim 1$

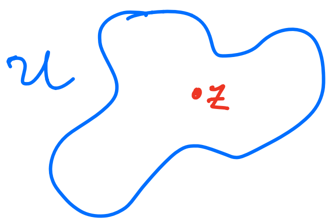
$$ds = \frac{2\,|dz|}{(1-|z|)(1+|z|)} = \frac{2}{1+|z|} \cdot \frac{|dz|}{1-|z|}$$

$$\approx \frac{|dz|}{d_E(z, \partial\mathbb{D})}$$

In fact: if $\mathcal{U}$ is simply connected open, proper subset of $\mathbb{C}$, then you can define a Poincaré metric on $\mathcal{U}$

$$\mathcal{U} \xrightarrow{\varphi} \mathbb{D}$$

$$\rho_{\mathcal{U}} := \varphi^* \rho_{\mathbb{D}}$$



$$ds \approx \frac{|dz|}{d_E(z, \partial\mathcal{U})}$$

Pf (Sketch)

① if $ds^2 = g_{11} dx^2 + 2g_{12}\, dx\, dy + g_{22}\, dy^2$ is invariant under $\mathrm{Aut}(\mathbb{D})$

then at $z=0$

$$ds^2 = g_{11}(dx^2 + dy^2)$$

(applying invariance by rotation)

$\Rightarrow$ ds is <u>CONFORMAL</u>

i.e $\exists\, \rho : \mathbb{D} \to \mathbb{R}^{\geq 0}$ s.t.

$$ds = \rho(z)\, dz$$

② if $ds = \rho(z)\, dz$ is invariant by $\gamma \in \mathrm{Aut}(\mathbb{D})$, then :

$z' = \gamma(z)$

$$\rho(\gamma(z))\,\gamma'(z)\,dz = \rho(z)\,dz$$

$$\rho(\gamma(z)) = \frac{\rho(z)}{\gamma'(z)} \quad \text{for all } z \in \mathbb{D}.$$

in upper half plane model

$$\rho(\gamma(z)) = \frac{\rho(z)}{\gamma'(z)} \quad \text{for all } z \in \mathbb{H}$$

$$\gamma(z) = a + bz \qquad , (b>0)$$

$$\rho(a+bz) = \frac{\rho(z)}{b} \qquad \forall a, \forall b>0, \ \forall z \in \mathbb{H}$$

$$\rho(a+bi) = \frac{1}{b}\rho(i) \overset{\rho(i)=1}{=} \frac{1}{b}$$

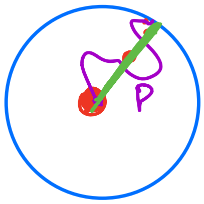
$$ds = \frac{dz}{\operatorname{Im}(z)} = \frac{dz}{y} \qquad (z = x+iy)$$

Lemma The disk $\mathbb{D}$ with the Poincaré metric is complete. Moreover:

① any path from a point in $\mathbb{D}$ to a point in $\partial\mathbb{D}$ has infinite length

② every two points are joined by a unique geodesic.

Pf



Pick path inside $\mathbb{D}$:

$$\int_P ds = \int_P \frac{2|dz|}{1-|z|^2} \geq \int_0^1 \frac{2\,dr}{1-r^2}$$

$$\int_0^1 \frac{2\,dr}{1-r^2} = \int \frac{dr}{1-r} + \int \frac{dr}{1+r} = \log\left(\frac{1+r}{1-r}\right) \xrightarrow{r\to 1} \infty$$

Thm (Schwarz, Pick, Ahlfors)

If $f: \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$\rho_{\mathbb{D}}(f(x), f(y)) \leq \rho_{\mathbb{D}}(x,y) \qquad \forall x,y \in \mathbb{D}$$

and if you have equality at some points then f must be an isometry.

Pf if $x = 0 = f(x)$, then Schwarz lemma
otherwise, apply $\gamma \in \mathrm{Aut}(\mathbb{D})$ to reduce to that situation (use invariance of metric).

Riemann sphere

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

carries a spherical metric

$$ds = \frac{2|dz|}{1+|z|^2}$$

has constant curvature $K \equiv +1$.

$$\mathrm{Aut}(\hat{\mathbb{C}}) = \left\{ z \mapsto \frac{az+b}{cz+d}, \begin{array}{l} a,b,c,d \in \mathbb{C} \\ ad-bc \neq 0 \end{array} \right\}$$

$$= SL_2(\mathbb{C}) / \{\pm I\} \simeq PSL_2(\mathbb{C})$$

Classification of elements of $PSL_2(\mathbb{R})$

$z \mapsto -\frac{1}{z}$ elliptic $\qquad |\mathrm{tr}\, A| < 2$

it has a fixed point inside $\mathbb{H}$

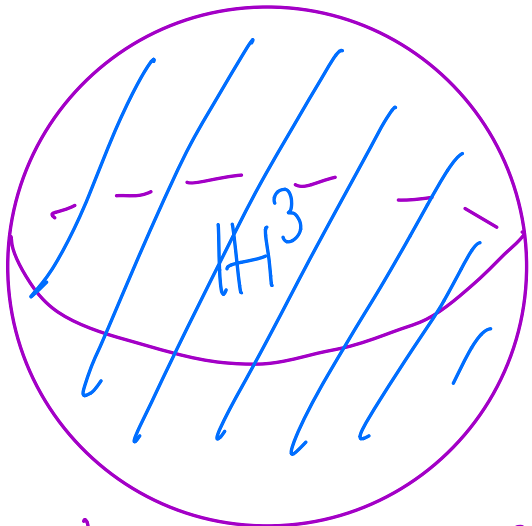
$z \mapsto z+1$ parabolic $\qquad |\mathrm{tr}\, A| = 2$

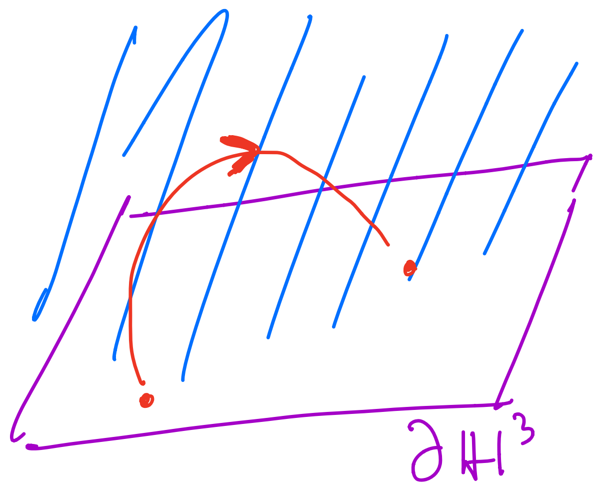
it has a fixed point on $\partial\mathbb{H}$

$z \mapsto \lambda z$ loxodromic/hyperbolic $\qquad |\mathrm{tr}\, A| > 2$

$(\lambda > 1)$ has two fixed points on $\partial\mathbb{H}$

<u>Recall</u> $PSL_2(\mathbb{C}) \simeq \mathrm{Isom}^+(\mathbb{H}^3)$





in $\mathbb{H}^3$ picture, a loxodromic element of $PSL_2(\mathbb{C})$ acts by translating <u>AND</u> rotating around the geodesic.

<u>Note</u>: (1) if an elt of $PSL_2(\mathbb{C})$ fixes 3 points of $\hat{\mathbb{C}}$, it is the identity.

$$z = \frac{az+b}{cz+d}$$

(2) <u>every</u> elt of $PSL_2(\mathbb{C})$ has <u>at least</u> one fixed point.

Cor.: The sphere does not cover any other Riemann surface.

Pf if S has $\hat{\mathbb{C}}$ as its univ. cover,

then $\hat{\mathbb{C}} \longrightarrow S$

$S = \hat{\mathbb{C}} / \Gamma$ where $\Gamma < \mathrm{Aut}(\hat{\mathbb{C}})$ discrete, fixed point free subgroup

But every elt has at least a fixed point, hence $\Gamma = \{1\}$.

EUCLIDEAN PLANE

$\mathbb{C}$ carries a flat metric

$|dz|$ $\qquad K \equiv 0$

$$\mathrm{Aut}(\mathbb{C}) = \{ z \mapsto az + b, \ a \neq 0, \ a, b \in \mathbb{C} \}$$

Riemann surfaces whose universal cover is $\mathbb{C}$

$$S = \mathbb{C} / \Gamma$$

$\Gamma < \mathrm{Aut}(\mathbb{C})$ discrete fixed point free

$z \mapsto az + b$ must have $\underline{a = 1}$

$z \mapsto z + b$

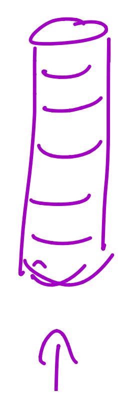
(1) $\Gamma = \{1\}$, $S = \mathbb{C}$

(2) $\Gamma = \langle z \mapsto z + 1 \rangle$

$$S = \mathbb{C} / z \mapsto z+1$$

cylinder carries a <u>flat metric</u>

$\mathbb{C}/\mathbb{Z}$

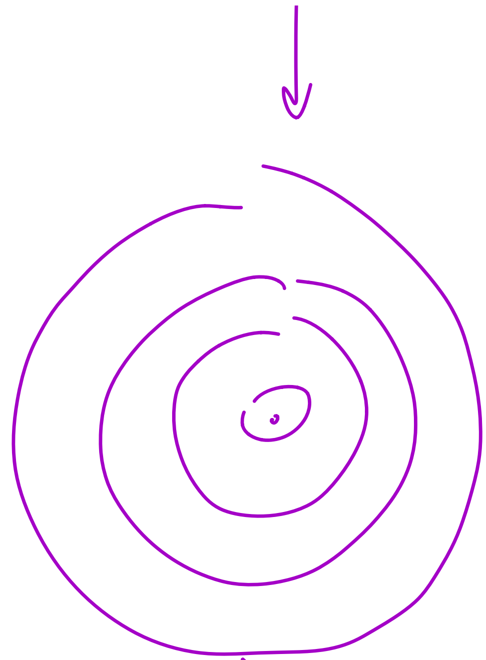


$\mathbb{C} \setminus \{0\}$

$$\mathbb{C} \xrightarrow{\exp(2\pi i z)} \mathbb{C}\setminus\{0\}$$

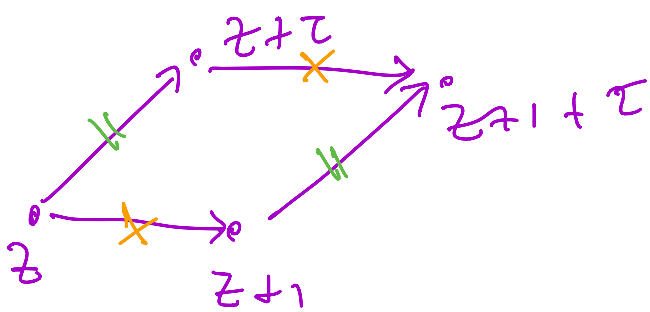
$$\exp(2\pi i(z+1)) = \exp(2\pi i z)$$

$$\mathbb{C}/\mathbb{Z} \xrightarrow{\sim} \mathbb{C}\setminus\{0\}$$



③ $\Gamma = \langle z \mapsto z+1,\ z \mapsto z+\tau \rangle$

$\operatorname{Im}\tau > 0$



$\mathbb{C}/\Gamma$ torus

Riemann surfaces whose universal cover is $\mathbb{D}$

(HYPERBOLIC SURFACES)

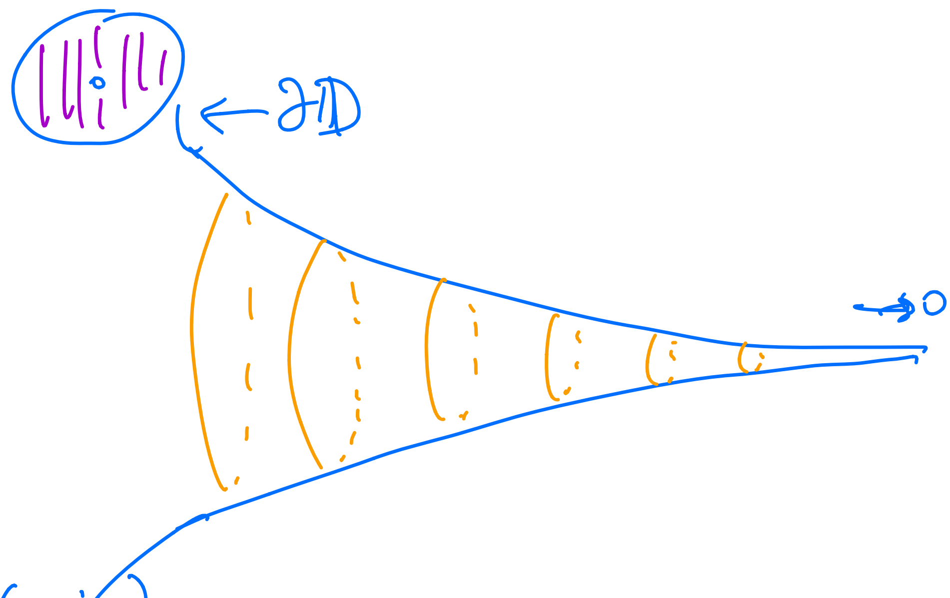
$S = \mathbb{D}/\Gamma \qquad \Gamma < \operatorname{Aut}(\mathbb{D}) \simeq PSL_2(\mathbb{R})$

fixed point free

All such Γ carry a Poincaré metric

$\Gamma = \langle z \mapsto z+1 \rangle$

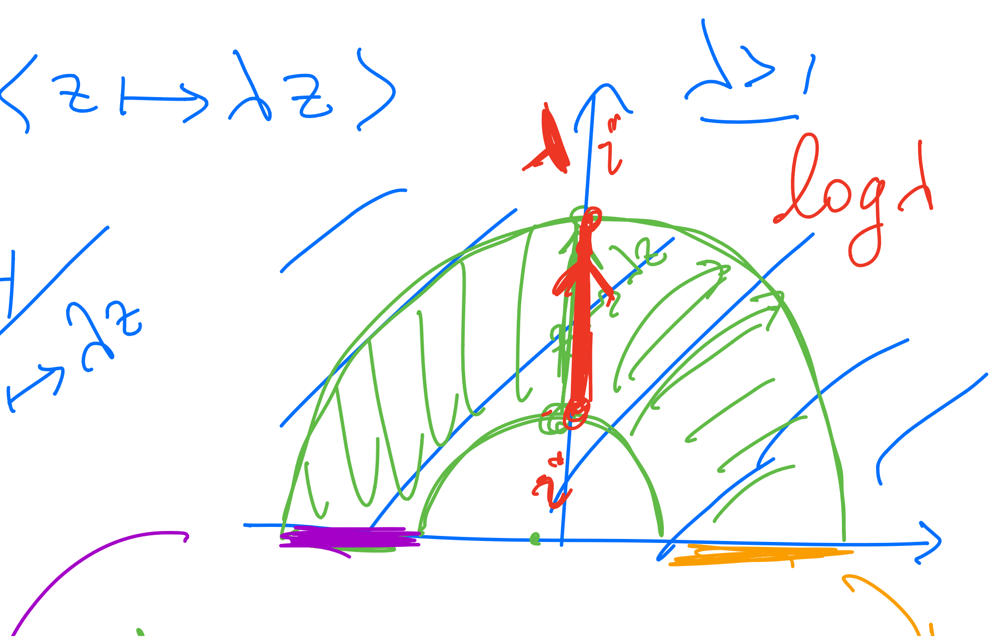
$\mathbb{H}/\mathbb{Z}$

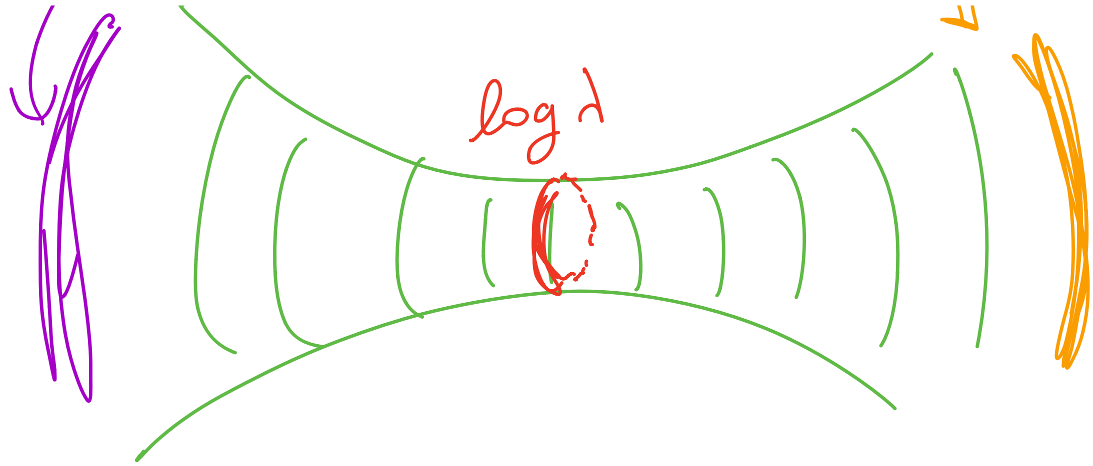


$\mathbb{H}/\mathbb{Z} \xrightarrow{\exp(2\pi i z)} \mathbb{D} \setminus \{0\}$

$\Gamma = \langle z \mapsto \lambda z \rangle$ $\quad \lambda > 1$

$\mathbb{H} / z \mapsto \lambda z$

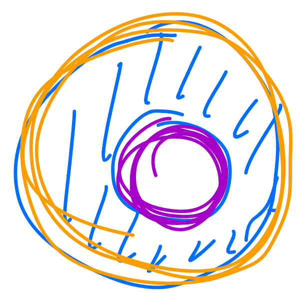




$$A_R = \{ z : 1 < |z| < R \}$$

$$\mathbb{H} \xrightarrow{\pi} A_R$$

$$z \mapsto \lambda z$$



$$\pi(z) = \pi(\lambda z)$$

$$\pi(z) = \exp\left(\frac{2\pi i}{\log \lambda} \log(z) \right)$$

$$2\pi i \frac{\log(\lambda z)}{\log \lambda} = 2\pi i + 2\pi i \frac{\log z}{\log \lambda}$$

σ σ

$\underline{z>0}$ $\pi(\mathbb{R}^+) = S^1$

$\underline{z<0}$ $\log(z) = i\pi + \log|z|$

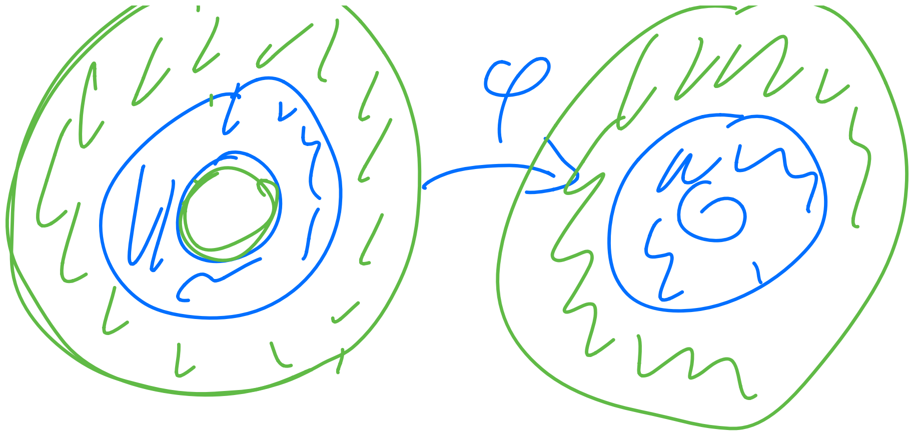
$$\pi(z) = \exp\left(\frac{2\pi i}{\log\lambda}\left(i\pi + \log|z|\right)\right) =$$

$$= \exp\left(\frac{-2\pi^2}{\log\lambda} + \frac{2\pi i \log|z|}{\log\lambda}\right)$$

$$R = \exp\left(\frac{-2\pi^2}{\log\lambda}\right)$$

$$d_{H}(i, \lambda i) = \int_1^\lambda \frac{dx}{x} = \log\lambda$$

$$A_s^t = \{ s < |z| < t \} \cong \{ 1 < |z| < \tfrac{t}{s} \}$$

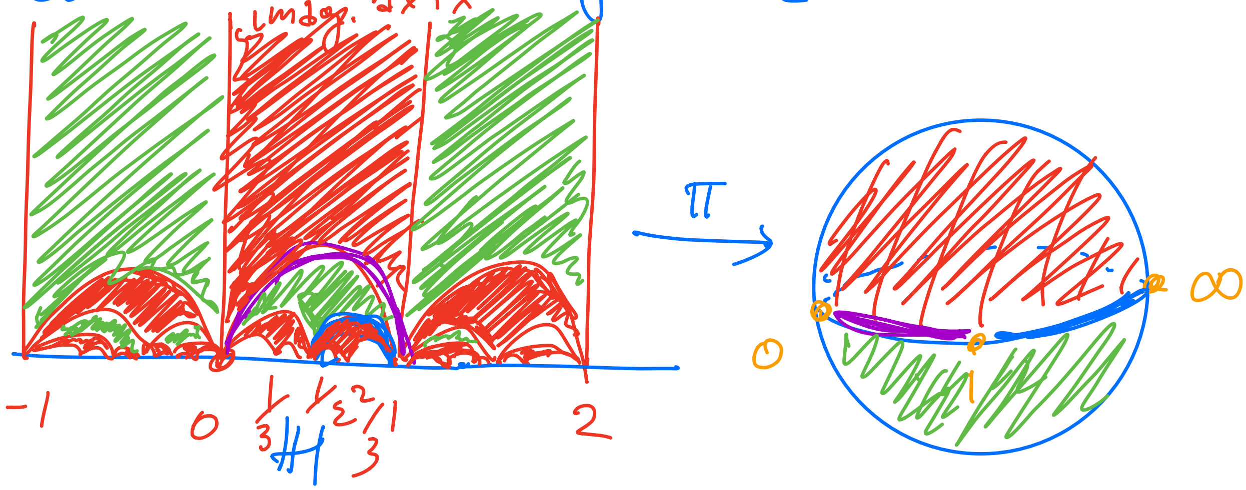


Hyperbolic Surfaces

$$\hat{\mathbb{C}} \setminus \{0,1,\infty\} \cong \mathbb{C} \setminus \{0,1\}$$

$$\mathbb{H} \xrightarrow{\pi} \mathbb{C} \setminus \{0,1,\infty\}$$

Consider action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$



$$\pi(\gamma(z)) = \pi(z) \quad \text{if } \gamma \in SL_2(\mathbb{Z})$$

Thm $\mathbb{H} / \Gamma \approx \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$